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LIOUVILLIAN SOLUTIONS OF FIRST ORDER ALGEBRAIC ORDINARY DIFFERENTIAL EQUATIONS

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Introduction

A *differential equation* (DE) is an equation that includes one or more unknown functions and their derivatives. If a DE contains an unknown function and its derivatives which depend on an independent variable x then it is called an *ordinary differential equation* (ODE). A DE is called *linear* if the relationship of the unknown function and its derivatives is linear; otherwise, it is called *nonlinear*. Most ODEs encountered in physics are linear, therefore, there are many approach for solving them. An idea of transforming nonlinear DEs into linear DEs and then solve the last ones may be a reasonable candidate. However, it works for only some cases. Therefore, studying independently the solutions of nonlinear DEs is necessary, and it also contains a lot of challenges. In this dissertation, we study liouvillian general solutions of first-order *algebraic ordinary differential equations* (AODEs) which is a fundamental problem in the theory of non-linear algebraic DEs.

A first-order AODE is a DE of the form $F(y, y') = 0$, where F is an irreducible polynomial in two variables with coefficients in $\mathbb{K}(x)$, $\mathbb{K}$ is an algebraically closed field of characteristic zero. Solving such DE is a problem of finding differentiable functions $y = y(x)$ satisfying $F(y(x), y'(x)) = 0$. If $y(x)$ belongs to $\mathbb{K}(x)$ (resp. an algebraic extension field of $\mathbb{K}(x)$), then it is called a *rational solution* (resp. an *algebraic solution*). If such a solution $y(x)$ belongs to a liouvillian extension of $\mathbb{K}(x)$, then it is called a *liouvillian solution*. A solution may contain an arbitrary constant. In this case, such a solution is called a *general solution*. For example, $y(x) = \exp(x^2 + c)$ is a *liouvillian general solution* of the first-order AODE $y' - 2xy = 0$.

First-order AODEs have been studied a lot, and there are many solution methods for special classes of such AODEs. The study of these AODEs can be dated back to the works of Fuchs [16] (1884). In [20] (1926), Ince presented an overall picture of ODEs. In

[30,31] (1970s), Matsuda classified differential function fields having no movable critical points up to isomorphism of differential fields. By focusing on particular solutions, in [29] (1913), Malmquist studied the class of first-order AODEs having transcendental meromorphic solutions, and Eremenko revisited later in [10] (1982). Applied Matsuda's theory, Eremenko in [11] (1998) presented a theoretical consideration on a degree bound for rational solutions which sheds light on the problem of finding such a solution's explicit form.

Finding the closed form solution of an ODE can be traced back to the works of Liouville (1830s) for the simplest ODE $y' = \alpha$, where $\alpha \in \mathbf{k}$ and $\mathbf{k}$ is a differential field of characteristic zero. If such an equation has a solution in some elementary differential extension field E of $\mathbf{k}$ having the same subfield of constants $\mathbb{K}$, then there exist constants $c_1, c_2, \ldots, c_n \in \overline{\mathbb{K}}$, elements $u_1, u_2, \ldots, u_n \in \overline{\mathbb{K}}\mathbf{k}$ and $v \in \mathbf{k}$ such that

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'.$$

In [44] (1968), Rosenlicht showed how Liouville theorem can be handled algebraically. For the algorithm consideration of such ODE, the pioneer work is due to Risch. In [41, 42] (1960s), Risch described a method to determine an *elementary integral* $\int u$ where u is an *elementary function*. To extend Risch's method, in [51, 52] (1970s), Singer studied *elementary solutions* of first-order AODEs. As a special result, there are necessary and sufficient conditions for the ODE $y' = R(y) \in \mathbb{C}(y)$ having an elementary solution. In [56] (2017), Srinivasan generalized this result to the case of liouvillian solutions with the same conditions. In [25] (1986), Kovacic presented an effective method to find liouvillian solutions of second order linear homogeneous ODEs. This work contains an algorithm for finding rational general solutions of a Riccati equation which is applicable to the works of Chen and Ma [7] (2005) and Vo et al. [57] (2018) for determining rational general solutions of first-order parametrizable AODEs. In [19] (1996), Hubert studied implicit general solutions of $F(y,y')=0$ by computing Gröbner bases. In [40] (1983), Prelle and Singer studied *elementary first integrals* $I(x,y)$ of the following system of ODEs

$$\frac{dx}{dz}=P(x,y);\ \frac{dy}{dz}=Q(x,y),\ \text{where } P(x,y),Q(x,y)\in\mathbb{C}[x,y].$$

Such a first integral induces a general solution $I(x,y)=c$ of the ODE $y'=\frac{Q(x,y)}{P(x,y)}$. In [55] (1992), Singer revisited this problem for finding *liouvillian first integrals* $I(x,y)$. In computational aspects, recently, Duarte and Da Mota, [9] (2021), presented an efficient method for computing liouvillian first integrals.

The starting point for the algebro-geometric method was algorithms introduced by the works of Feng and Gao in [14,15] (2000s). These algorithms decide whether or not an autonomous first-order AODE, $F(y,y')=0$, has a rational general solution and computes it if there is any. The key point is that a rational solution of such an AODE induces a proper rational parametrization of the corresponding algebraic curve, from that, we find a reparametrization such that the second component is the derivative of the first one. The existence of a proper parametrization can be decided by the works of Sendra and Winkler [49] (2001). From that, a rational general solution can be deduced.

Using the ideas of Feng and Gao in [14,15], several generalizations have been investigated since then. There are (not exhausted) notable works. In [7], by means of rational parametrizations, Chen and Ma reduced the problem of determining rational general solutions of a first-order parametrizable AODE to the case of solving a Riccati equation. In here, the method in [25] for finding rational general solutions (of a Riccati equation) can be applied. This work is not complete due to the rational forms of the rational parametrizations over a rational function field are required. In [33, 34] (2010s), Ngo and Winkler introduced a method based on parametrizations of surfaces for finding rational general solutions of such parametrizable AODEs. In [57], by determining an optimal parametrization of an algebraic curve over a rational function field, Vo et al. overcame the missing steps of [7] and obtained a decision algorithm of finding strong rational general solutions of first-order AODEs. A summarization and more aspects of the algebro-geometric method can be found in Sebastian et al. [12] (2023).

In this dissertation, we inherit and extend the works by Feng and Gao [14, 15], Srinivasan [56], and Vo et al. [57] for determining liouvillian general solutions of first-order AODEs. In particular, the dissertation contributes the following results.

- Define rational liouvillian solutions (Definition 2.2.3) and give Algorithm `RatLiouSol` in Section 2.4 for finding such rational liouvillian solutions of first-order autonomous AODEs.
- Show that liouvillian solutions (which include the class of algebraic solutions) of a first-order autonomous AODE of genus zero must be rational liouvillian solutions (Lemma 3.2.2) and give Algorithm `LiouSolAut` in Section 3.3 for determining and classifying such liouvillian solutions in algebraic and transcendental cases.
- Give Algorithm `LiouSol` in Section 4.1.2 for finding liouvillian solutions of first-order AODEs of genus zero (included autonomous and non-autonomous cases).
- Define power transformations (Definition 4.2.1) and propose Algorithm `RedPol` in Section 4.2.2 to obtain reduced forms of first-order AODEs. This result leads to a method for finding liouvillian solutions of first-order AODEs of positive genera in case their reduced forms are of genus zero (Section 4.2.3).
- Transform the problem of solving first-order AODEs with liouvillian coefficients into the case of solving an AODE (4.1) by means of change of variables (Section 4.4).

This dissertation summarizes our works in the last three years and give short description of our future research. The dissertation is organized as follows.

Chapter 1 presents basic materials in differential algebra and algebraic geometry. It also contains the main tools using regularly in the dissertation.

In Chapter 2, we define rational liouvillian solutions of first-order autonomous AODEs. Using the properties of rational parametrizations of algebraic curves, we give necessary and sufficient conditions for a first-order autonomous AODE to have rational liouvillian solutions. Based on this, we present an algorithm for determining rational liouvillian solutions of first-order autonomous AODEs.

In Chapter 3, we apply the theory of fields of algebraic functions of one variable to show that a liouvillian solutions of a first-order autonomous AODE of genus zero, if there exists, must be a rational liouvillian solution. By using Sylvester resultant, the forms of the liouvillian solutions can be described in algebraic relations. These results lead to an algorithm for determining the existence of such liouvillian solutions.

In Chapter 4, we study liouvillian solutions of first-order AODEs of genus zero via their associated ODEs by means of rational parametrizations. Using the theory of fields of algebraic functions of one variable, we prove that the property of having a liouvillian general solution of the two above DEs are the same. This result covers the autonomous case considered in Chapter 3. If first-order AODEs are of positive genera, there is an approach for solving them. First, we give an algorithm to compute a reduced form of a given AODE by means of power transformations. From that, we give a method for finding liouvillian solutions of first-order AODEs of positive genera whose reduced forms are of genus zero. Finally, we study the problem of solving first-order AODEs with coefficients in a liouvillian extension of $\mathbb{C}(x)$ by means of change of variables.

Chapter 1

Preliminaries

The content of differential algebra and algebraic geometry which are necessary for the dissertation can be found in the standard textbooks such as [4, 24, 43] and [8, 27, 50, 59], respectively.

1.1 Differential algebra

Definition 1.1.1. Let $\mathbf{k}$ be a field of characteristic zero. A *derivation* of the field $\mathbf{k}$, denote by $'$, is an operation of $\mathbf{k}$ that satisfies the two following items:

1. $(a+b)' = a' + b'$
2. $(ab)' = a'b + ab'$

for every $a, b \in \mathbf{k}$. A field $\mathbf{k}$ equipped with a derivation $'$ is called a *differential field*. An element $a \in \mathbf{k}$ is called a *constant* if $a' = 0$.

Definition 1.1.2. A field extension E of $\mathbf{k}$ is called a *differential field extension* of $\mathbf{k}$ if and only if the derivation of E restricted to $\mathbf{k}$ coincides with the derivation of $\mathbf{k}$.

1.2 Plane algebraic curves

Definition 1.2.1. Let K be an algebraically closed field of characteristic zero. A subset $\mathcal{C} \subset \mathbb{A}^2(K)$ is called an *affine algebraic curve* (a curve, for briefly) if there is a non-constant irreducible polynomial $F \in K[X, Y]$ such that $\mathcal{C} = \mathcal{V}(F)$. Such F is called the *defining polynomial* of $\mathcal{C}$. By abuse of notation, we sometime call $F(x, y) = 0$ an affine algebraic curve.

1.3 Fields of algebraic functions of one variable

Definition 1.3.1. Let K be an algebraically closed field of characteristic zero. A field $L \supset K$ is called *a field of algebraic functions of one variable* over K if it satisfies the following condition: L contains an element x which is transcendental over K, and L is algebraic of finite degree over $K(x)$.

1.4 Rational functions on algebraic curves

The content of this section can be found in [26, Chapter 4].

1.5 Preparation

Definition 1.5.1. Let $F(y, y') = 0$ be a first-order AODE over K. The algebraic curve $F(y, w) = 0$ where $F(y, w) \in K[y, w]$ is said to be the *corresponding algebraic curve* of the AODE $F(y, y') = 0$.

1.5.1 Associated fields of algebraic functions

Definition 1.5.2. Assume that L is a field of algebraic functions over K, then there are $\eta, \xi \in L$ such that $L = K(\eta, \xi)$, where η is transcendental over K and ξ is algebraic over $K(\eta)$. The function field $L = K(\eta, \xi)$ is called an *associated field of algebraic functions* of the affine algebraic curve $\mathcal{C}$ defined by irreducible polynomial F if $F(\eta, \xi) = 0$. Such $\mathcal{C}$ is called the *affine algebraic curve model* of the function field L.

Lemma 1.5.3. *[37, Lemma 2.7] If η is a solution of the AODE $F(y, y') = 0$ which is transcendental over K, then $K(\eta, \eta')$ is an associated field of algebraic functions of the corresponding algebraic curve $\mathcal{C}$ defined by $F(y, w)$. In addition, if $\mathcal{C}$ is of genus zero then its associated field of algebraic functions $K(\eta, \eta')$ is of the form $K(t)$.*

1.5.2 Rational parametrizations

Definition 1.5.5. A *rational parametrization* of an algebraic curve $\mathcal{C}$ defined by an irreducible polynomial $F(y, w)$ is a pair of rational functions $\mathcal{P}(t) = (r(t), s(t)) \in K(t)^2$ such that the two following items hold.

1. For almost all t_0 the point $\mathcal{P}(t_0) = (r(t_0), s(t_0)) \in \mathcal{C}$.

2. For almost all point $(x_0, y_0) \in \mathcal{C}$ there exists $t_0 \in K$ such that $\mathcal{P}(t_0) = (x_0, y_0)$.

An algebraic curve $\mathcal{C}$ is said to be *rational* or a *rational curve* if it admits a rational parametrization $\mathcal{P}(t)$. Moreover, if t_0 is unique then such $\mathcal{P}(t)$ is said to be *proper* or a *proper parametrization* of $\mathcal{C}$.

Chapter 2

Rational liouvillian solutions of first-order autonomous AODEs

This chapter considers first-order autonomous AODEs

$$F(y, y') = 0. \tag{2.1}$$

2.1 Solving first-order AODEs by parametrizations

Theorem 2.1.4. *[14, Theorem 5] Let $y = r(x), w = s(x)$ be a proper parametrization of $F(y, w) = 0$ where $r(x), s(x) \in \bar{\mathbb{Q}}(x)$. Then $F(y, y') = 0$ has a rational general solution if and only if there are the following relations*

$$ar'(x) = s(x) \quad \text{or} \quad a(x-b)^2 r'(x) = s(x) \tag{2.4}$$

where $a, b \in \bar{\mathbb{Q}}$ and $a \neq 0$. If one of the above relations is true, then replacing x by $a(x+c)$ (or $b - \dfrac{1}{a(x+c)}$) in $y = r(x)$, we obtain a rational general solution of $F = 0$, where c is an arbitrary constant.

2.2 Rational liouvillian solutions

Definition 2.2.2. [35, Definition 2.7] Let E be a liouvillian extension of $\mathbb{C}$ and let $t \in E \backslash \mathbb{C}$. t is called *rational liouvillian element* over $\mathbb{C}$ if $t' \in \mathbb{C}(t)$.

Definition 2.2.3. [35, Definition 2.8] A solution $y = r(t)$ of an AODE $F(y, y') = 0$ (2.1) is called a *rational liouvillian solution* over $\mathbb{C}$ if it is of the form

$$r(t) = \frac{a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0}{b_m t^m + b_{m-1} t^{m-1} + \cdots + b_1 t + b_0}$$

where $m, n \in \mathbb{N}$, $a_i, b_j \in \mathbb{C}$, and t is a rational liouvillian element over $\mathbb{C}$.

2.3 Main results

Lemma 2.3.1. [35, Lemma 3.1] *If an AODE $F(y, y') = 0$ (2.1) has a non-constant rational liouvillian solution then $F(y, w) = 0$ is a rational curve.*

Lemma 2.3.4. [35, Lemma 3.2] *Let $(r_1(t), s_1(t))$ be a proper parametrization of the algebraic curve $F(y, w) = 0$. If the first-order AODE $F(y, y') = 0$ has a non-constant rational liouvillian solution then $\dfrac{\frac{dr_1}{dt}}{s_1(t)}$ must be of the form $\dfrac{dz}{dt}$ or $\dfrac{\frac{dz}{dt}}{az}$, where $z \in \mathbb{C}(t)$ and $a \in \mathbb{C} \backslash 0$.*

Lemma 2.3.5. [35, Lemma 3.3] *Let $(r(t), s(t))$ be a proper parametrization of $F(y, w) = 0$. Set $h(t) = \dfrac{\frac{dr}{dt}}{s(t)}$, we have two cases:*

1. If there is an element $z(t) \in \mathbb{C}(t)$ such that $h(t) = \frac{dz}{dt}$, then by setting $z(t) = x$, we obtain $r(t)$ is a rational liouvillian solution of $F(y, y') = 0$.

2. If there is an element $z(t) \in \mathbb{C}(t)$ such that $h(t) = \frac{\frac{dz}{dt}}{az}$ for some non-zero $a \in \mathbb{C}$, then by setting $z(t) = \exp ax$, we obtain $r(t)$ is a rational liouvillian solution of $F(y, y') = 0$.

Lemma 2.3.6. [35, Lemma 3.4] *Let $F(y, w) = 0$ be a rational curve. Suppose that $(r_1(t), s_1(t))$ and $(r_2(t), s_2(t))$ are two different proper rational parametrizations $F(y, w) = 0$. Then the two differential equations*

$$t' = \frac{s_1(t)}{\frac{dr_1(t)}{dt}} \quad \text{and} \quad t' = \frac{s_2(t)}{\frac{dr_2(t)}{dt}}$$

have the same liouvillian solvability.

Theorem 2.3.7. [35, Theorem 3.1] *A first-order autonomous AODE $F(y, y') = 0$ has a non-constant rational liouvillian solution if and only if the algebraic curve $F(y, w) = 0$ is rational and for every proper parametrization $(r(t), s(t))$, there exists $z(t) \in \mathbb{C}(t)$ such that $\frac{\frac{dr}{dt}}{s(t)}$ is of the form $\frac{dz}{dt}$ or $\frac{\frac{dz}{dt}}{az}$ for some non-zero $a \in \mathbb{C}$. In the first case, let $z(t) = x$, and in the second case, let $z(t) = \exp ax$, where $x' = 1$, then $r(t)$ is a rational liouvillian solution of $F(y, y') = 0$.*

2.4 An algorithm and examples

Algorithm `RatLiouSol`

Input: An algebraic curve $F(y, w) = 0$.

Output: A rational liouvillian solution of $F(y, y') = 0$ if any.

---

1. If the algebraic curve $F(y, w) = 0$ is not rational, then **return** "$F(y, y') = 0$ does not have a rational liouvillian solution". Else,

2. Compute a proper parametrization $(r(t), s(t))$ of $F(y, w) = 0$ and set $h(t) = \frac{\frac{dr}{dt}}{s(t)}$.

3. If $h(t)$ is not satisfied the cases of Theorem 2.3.7, then **return** "$F(y, y') = 0$ has no rational liouvillian solution". Else,

4. If $h(t) = \frac{dz}{dt}$ where $z(t) \in \mathbb{C}(t)$, then setting $z(t) = x$. There are some cases.

   (a) If $h(t) = \frac{1}{a} \in \mathbb{C}$, then $z(t) = \frac{t}{a} = x$. So $t = g(x) = ax$, and $y = r(ax)$ is a rational solution. It also gets $r(a(x + c))$ is a rational general solution.

   (b) If $h(t) = \frac{1}{a(t-b)^2}$, then $z(t) = \frac{-1}{a(t-b)} = x$, so $t = g(x) = b - \frac{1}{ax}$. Hence $y = r\left(b - \frac{1}{ax}\right)$ is a rational solution. It also gets $r\left(b - \frac{1}{a(x+c)}\right)$ is a rational general solution.

   (c) If $t = g(x)$ and both cases (a) and (b) do not occur, then $F(y, y') = 0$ has a radical solution $r(g(x))$. In this case, $r(g(x + c))$ is a radical general solution.

   (d) If there is not an explicit function $g(x)$ such that $t = g(x)$, then $r(t)$ is a rational liouvillian solution which is not a radical solution.

5. If $h(t) = \frac{\frac{dz}{dt}}{az}$ with $z(t) \in \mathbb{C}(t)$, then we set $z(t) = \exp ax$. Assume $t = g(x)$, then we get $r(g(x))$ is a rational liouvillian solution of $F(y, y') = 0$ which is not algebraic. In this case, $r(g(x + c))$ is a rational liouvillian general solution.

Chapter 3

Liouvillian solutions of first-order autonomous AODEs of genus zero

3.1 Sylvester resultant

Let $\mathcal{P}(t) = \left(\frac{m(t)}{n(t)}, \frac{p(t)}{q(t)}\right)$, where $m(t), n(t), p(t), q(t) \in \mathbb{C}[t]$, and consider the following polynomials

$$G_1^{\mathcal{P}}(s,t) = m(s)n(t) - n(s)m(t), \quad G_2^{\mathcal{P}}(s,t) = p(s)q(t) - q(s)p(t)$$

as well as the polynomials

$$H_1^{\mathcal{P}}(t,x) = x.n(t) - m(t), \quad H_2^{\mathcal{P}}(t,y) = y.q(t) - p(t).$$

Lemma 3.1.2. *[50, Lemma 4.6] Let $\mathcal{C}$ be a rational algebraic curve defining by $F(x,y)$, and $\mathcal{P}(t)$ be a rational parametrization of $\mathcal{C}$. Then there exists $h \in \mathbb{N}$ such that*

$$res_t(H_1^{\mathcal{P}}(t,x), H_2^{\mathcal{P}}(t,y)) = (F(x,y))^h.$$

3.2 Main results

Lemma 3.2.2. [36, Lemma 3.2] *Let $F(y, w) = 0$ be a rational curve. Assume that η is a liouvillian solution of $F(y, y') = 0$ over $\mathbb{C}$ then η is a rational liouvillian solution over $\mathbb{C}$.*

Theorem 3.2.3. [36, Theorem 3.3] *Let $F(y, w) = 0$ be a rational curve. The first-order autonomous AODE $F(y, y') = 0$ has a liouvillian solution over $\mathbb{C}$ if and only if for every proper parametrization $(r(t), s(t))$ of $F(y, w) = 0$, there is a $z(t) \in \mathbb{C}(t)$ such that the associated function $h(t)$ is either of the form $\frac{dz}{dt}$ or $\frac{\frac{dz}{dt}}{az}$ for some non-zero $a \in \mathbb{C}$. In the first case, let $z(t) = x$, and in the second case, let $z(t) = \exp(ax)$, then $r(t)$ is a liouvillian solution of $F(y, y') = 0$.*

Lemma 3.2.5. [36, Lemma 3.5] *Assume that $F(y, w) = 0$ has a proper parametrization $(r(t), s(t))$ such that $h(t)$ is of the form $\frac{dz}{dt}$, $z(t) \in \mathbb{C}(t)$. If $F(y, w) = 0$ has another parametrization $(r_1(t), s_1(t))$, then $h_1(t)$ is of the form $\frac{dz_1}{dt}$, $z_1(t) \in \mathbb{C}(t)$.*

Lemma 3.2.6. *Let $F(y, w) = 0$ be a rational curve. If $G(x, y) = 0$ is an algebraic solution of $F(y, y') = 0$, then the genus of $G(x, y) = 0$ is zero.*

Theorem 3.2.8. [36, Theorem 3.8] *Let $F(y, w) = 0$ be a rational curve. The differential equation $F(y, y') = 0$ has an algebraic solution $G(x, y) = 0$ if and only if the associated function $h(t)$ is of the form $\frac{dz}{dt}$, where $z(t) \in \mathbb{C}(t)$. If the solution exists, the defining polynomial $G(x, y)$ can be determined by its parametrization.*

Theorem 3.2.9. [36, Theorem 3.9] *Let $F(y, w) = 0$ be a rational curve. Assume that η is a non-algebraic liouvillian solution of $F(y, y') = 0$. There are a non-zero element $a \in \mathbb{C}$ and an irreducible polynomial G such that $G(\exp(ax), \eta) = 0$. In other words, η is algebraic over $\mathbb{C}(\exp(ax))$.*

3.3 An algorithm and applications

Algorithm `LiouSolAut`

Input: A rational algebraic curve $F(y, w) = 0$.

Output: A liouvillian general solution of $F(y, y') = 0$ if any.

---

1. Compute a proper parametrization $(r(t), s(t))$ of the algebraic curve $F(y, w) = 0$ and the associated function $h(t) = \frac{\frac{dr}{dt}}{s(t)}$.

2. If $h(t) = \frac{dz}{dt}$ with $z(t) \in \mathbb{C}(t)$, then set $z(t) = x$ and $\mathcal{P}(t) = (z(t), r(t))$. Set $G(x, y)$ is the square-free part of

$$res_t(H_1^{\mathcal{P}}(t, x), H_2^{\mathcal{P}}(t, y))$$

in Lemma 3.1.2, then $G(x, y) = 0$ is an algebraic solution. Hence, an algebraic general solution of the given equation is $G(x + c, y) = 0$.

3. If $h(t) = \frac{\frac{dz}{dt}}{az}$ with $z(t) \in \mathbb{C}(t)$, then set $z(t) = \exp(ax) = u$. Set $\mathcal{P}(t) = (z(t), r(t))$, by processing the same way of the case (2.), we obtain $G(u, y) = 0$ is a non-algebraic liouvillian solution. Then $G(\exp(a(x + c)), y) = 0$ is a liouvillian general solution.

4. Otherwise, the algorithm terminates, and $F(y, y') = 0$ has no liouvillian solution.

In application, let $P(y) \in \mathbb{C}[y]$ be a polynomial of degree 3. Consider the AODE

$$y'^2 = P(y). \tag{3.1}$$

Proposition 3.3.8. [36, Proposition 4.7 and Remark 4.8] *The AODE (3.1) has a liouvillian solution over $\mathbb{C}$ if an only if $P(y) = 0$ has repeated roots.*

Chapter 4

Liouvillian solutions of first-order AODEs

This chapter considers a first-order AODE of the form

$$F(Y, Y') = 0, \tag{4.1}$$

where F is an irreducible polynomial of $\mathbb{C}(z)[y, w]$.

4.1 Liouvillian solutions of first-order AODEs of genus zero

4.1.1 Associated differential equations

Find a solution of an AODE (4.1) in case of it has genus zero via a proper parametrization $\mathcal{P}(t) = (u(t), v(t))$ induces a problem of considering the following differential equation

$$t'(z) = \frac{v(t) - \dfrac{\partial u(t)}{\partial z}}{\dfrac{\partial u(t)}{\partial t}}. \tag{4.3}$$

Definition 4.1.1. The ODE (4.3) is called an *associated differential equation* of the AODE (4.1) respect to a proper parametrization $\mathcal{P}(t) = (u(t), v(t))$.

Lemma 4.1.2. *[37, Lemma 3.3] Let $P(t)$ and $\tilde{\mathcal{P}}(t)$ be two proper parametrizations of F. There is a change of variables $s = \dfrac{\alpha t + \beta}{\gamma t + \delta}$, where $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{C}(z)}$, $\alpha\delta - \beta\gamma \neq 0$, between the two associated equations of F respect to $\mathcal{P}(t)$ and $\tilde{\mathcal{P}}(t)$.*

4.1.2 Main results and an algorithm

Theorem 4.1.3. *[37, Theorem 3.4] An AODE (4.1) of genus zero has a liouvillian general solution if and only if so does its associated ODE (4.3) respect to a certain proper parametrization $\mathcal{P}(s)$.*

From [57, Theorem 4.3], there is an optimal parametrization $\mathcal{P}(t)$ such that ODE (4.3) is of the form

$$t' = \frac{dt}{dz} = f(z, t) \in \mathbb{C}(z, t). \tag{4.7}$$

Therefore, without loss of generality, we may consider an associated ODE of the form (4.7) when solving the original AODE (4.1).

Algorithm LiouSol

Input: A first-order AODE $F(Y, Y') = 0$ (4.1) of genus zero.

Output: A liouvillian general solution of (4.1) if any.

1. Find an optimal proper parametrization of $F(y, w) = 0$

$$\mathcal{P}(t) = (u(t), v(t)) \in (\mathbb{C}(z, t))^2.$$

2. Compute the associated ODE (4.7) respect to $\mathcal{P}(t)$.
3. If the ODE (4.7) has a liouvillian general solution $t(z)$, then **return** "$Y(z) = u(t(z))$ is a liouvillian general solution of (4.1)".
4. Else, **return** "(4.1) has no liouvillian general solution".

4.1.3 An investigation of first-order ODEs (4.7) and examples

Proposition 4.1.5. [37, Proposition 3.6] *If the ODE (4.7) is of the form*

$$t' = b(z)t + c(z), \tag{4.8}$$

where $b(z), c(z) \in \mathbb{C}(z)$, then it always has a liouvillian solution.

Proposition 4.1.7. [37, Proposition 3.8] *Assume that the ODE (4.7) is of the form a Riccati equation*

$$t' = a(z)t^2 + b(z)t + c(z), \tag{4.9}$$

where $a(z), b(z), c(z) \in \mathbb{C}(z)$ and $a(z) \neq 0$. Then we can determine if it has a liouvillian solution or not.

Proposition 4.1.8. [37, Proposition 3.9] *If the ODE (4.7) is autonomous, we can determine if it has a liouvillian solution or not.*

4.2 Power transformations and their applications

4.2.1 Power transformations

Definition 4.2.1. [37, Definition 4.1] A *power transformation* is a transformation of the form

$$u = Y^n, u' = nY^{n-1}Y', 2 \leqslant n \in \mathbb{N}. \tag{4.19}$$

Let k_0 be the lowest degree of the non-zero homogeneous component of G, then by putting (4.19) into G we obtain

$$\begin{aligned} G(u,u') &= Y^{(n-1)k_0} \sum_{k=k_0}^{d} \sum_{i+j=k} c_{ij} n^j Y^{(n-1)(k-k_0)} Y^i Y'^j \\ &= Y^{(n-1)k_0} F(Y,Y'). \end{aligned} \tag{4.22}$$

Lemma 4.2.2. *[37, Lemma 4.2] Let $G(u,u')$ and $F(Y,Y')$ are two polynomials over $\mathbb{C}(z)$. If there is a power transformation (4.19) such that the formula (4.22) is satisfied, then the followings hold.*

1. *For each $k \geqslant k_0$, the polynomial*

   $$F_{n_k}(Y,Y') = \sum_{i+j=k} c_{ij} n^j Y^{(n-1)(k-k_0)} Y^i Y'^j$$

   is homogeneous of degree $n_k = n(k-k_0)+k_0$, and n_{k_0} is the lowest degree among the non-zero homogeneous components of F. Moreover, $n_{k_0} = k_0$.

2. *Let n_{k_1} and n_{k_2} be the degrees of two different homogeneous components of F. Then n is a common divisor of $(n_{k_1} - n_{k_0})$ and $(n_{k_2} - n_{k_0})$.*

3. *If F is an irreducible polynomial then so is G. In this case, if F has genus zero then the genus of G is zero too. Moreover, the reverse of these two properties are not true.*

4.2.2 Reduced forms by power transformations

Definition 4.2.3. [37, Definition 4.3] Let $F(Y, Y')$ be an irreducible polynomial. Let HD_F be the set of degrees of the non-zero homogeneous components of F, and let $k_0 = n_{k_0}$, see (1.) in Lemma 4.2.2, be the smallest element of HD_F. We define the set

$$\mathbb{D}_F = \{n \geqslant 2 \mid n \text{ is a common divisor of all } (m - k_0) \text{ for } m \in \mathrm{HD}_F\}. \tag{4.23}$$

Suppose that $\mathbb{D}_F \neq \varnothing$, and let $n \in \mathbb{D}_F$. We say such n *induces* a transformation of the form (4.19) if there is an irreducible polynomial $G(u, u')$ such that the formula (4.22) is satisfied. In this case, we say F is *transformed from* G by the transformation (4.19) respect to n. We define the set

$$\mathbb{P}_F = \{n \in \mathbb{D}_F \mid n \text{ induces a transformation (4.19)}\}. \tag{4.24}$$

Clearly that, $\mathbb{P}_F \subseteq \mathbb{D}_F$. If $\mathbb{D}_F = \varnothing$, then $\mathbb{P}_F$ is an empty set too.

Lemma 4.2.4. *[37, Lemma 4.4] Let $F(Y, Y')$ be an irreducible homogeneous polynomial then $\mathbb{D}_F$ is an infinite set. Moreover, $\mathbb{D}_F$ coincides with $\mathbb{P}_F$.*

Lemma 4.2.5. *[37, Lemma 4.5] Let $F(Y, Y')$ be an irreducible non-homogeneous polynomial, then $\mathbb{D}_F$ is either a finite set or an empty one. Moreover, $\mathbb{D}_F$ is different from $\mathbb{P}_F$.*

Definition 4.2.6. [37, Definition 4.6] An irreducible non-homogeneous polynomial F is called a *reduced form* if $\mathbb{P}_F = \varnothing$. Otherwise, F is called a non-reduced form.

Theorem 4.2.8. [37, Theorem 4.8] *Let F be a non-reduced form. Let n be the greatest element of $\mathbb{P}_F$ and G be an irreducible polynomial such that F is transformed form G respect to the power transformation induced by n. Then G is of a reduced form.*

Algorithm `RedPol`

Input: An irreducible non-homogeneous polynomial $F(Y, Y')$.

Output: The reduced form of F and the transformation (4.19) if any.

1. Rewrite F in non-zero homogeneous components to find HD_F.
2. Find k_0 and compute $\mathbb{D}_F$.
3. Determine $\mathbb{P}_F$.
4. If $\mathbb{P}_F = \varnothing$, then **return** "$F(Y, Y')$ is of reduced form and there is no power transformation (4.19)".
5. Else, let $n = \max \mathbb{P}_F$, then **return** "The reduced form $G(u, u')$ and the power transformation (4.19) respect to n".

Let $F(Y, Y')$ be defining polynomial of the first-order AODE (4.1) then its reduced form $G(u, u')$ (obtained by Algorithm `RedPol`) can be seen as defining polynomial of the AODE

$$G(u, u') = 0. \tag{4.25}$$

Theorem 4.2.11. [37, Theorem 4.11] *Suppose that $F(Y, Y') = 0$ (4.1) is transformed from a reduced AODE $G(u, u') = 0$ (4.25) by a power transformation (4.19) (respect to $n \geqslant 2$). Then (4.1) has a liouvillian solution if and only if so does (4.25). Moreover, if η is a liouvillian solution of (4.25), then there is a liouvillian solution ξ of (4.1) which satisfies*

$$Y^n - \eta = 0. \tag{4.26}$$

4.2.3 Applications

First-order AODEs of positive genera whose reduced form are first-order AODEs of genus zero can be solved by our method, see [18, Example 6] or [21, I·482, 485, 487, 504, 509, 541, 542, 543, 544].

4.3 Möbius transformations

A *Möbius transformation* is a transformation of the form

$$u = \frac{\alpha Y + \beta}{\gamma Y + \delta}, u' = \left(\frac{\alpha Y + \beta}{\gamma Y + \delta}\right)', \tag{4.34}$$

where $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{C}(z)}$, $\alpha\delta - \beta\gamma \neq 0$.

Theorem 4.3.3. [38, Theorem 3.1] *Assume that F is equivalent to G. Then F has a liouvillian solution if and only if so does G. In the affirmative case, the correspondence of such solution is one to one.*

4.4 Liouvillian solutions of first-order AODEs with liouvillian coefficients

Consider differential equation

$$\tilde{F}(y, y') = 0, \tag{4.40}$$

where y is a function of x and $\tilde{F} \in E[y, w]$, i.e. first-order AODEs with the coefficients in a liouvillian extension E of $\mathbb{C}(x)$. Assume that there is a change of variable

$$z = \varphi(x), \tag{4.41}$$

such that it turns an AODE (4.40) into an AODE (4.1), i.e.

$$\tilde{F}(y, y') = F(Y, Y') = 0,$$

where $F \in \mathbb{C}(z)[y, w]$. If $Y(z)$ is a liouvillian solution of (4.1), then

$$y(x) = Y \circ \varphi(x)$$

is a liouvillan solution of (4.40).

In case of considering transcendental coefficients, we refer to [4, Chapter V]. For the case of radical coefficients, we refer to [5].

Conclusion and future work

We have considered the class of first-order AODEs and studied their liouvillian solutions. Several methods have been proposed to attack the problem of finding these solutions for a first-order AODE. This dissertation has achieved the following main results.

1. We define a rational liouvillian solution (Definition 2.2.3) and give Algorithm `RatLiouSol` in Section 2.4 for finding rational liouvillian solutions of first-order autonomous AODEs.

2. We prove that liouvillian solutions (which include the class of algebraic solutions) of a first-order autonomous AODE of genus zero must be rational liouvillian solutions (Lemma 3.2.2) and give Algorithm `LiouSolAut` in Section 3.3 for finding and classifying such a liouvillian solution in algebraic and transcendental cases.

3. We propose an algorithm (Algorithm `LiouSol` in Section 4.1.2) for determining liouvillian solutions of first-order AODEs of genus zero (included autonomous and non-autonomous cases).

4. We define power transformations (Definition 4.2.1) and give Algorithm `RedPol` in Section 4.2.2 to obtain the reduced form of a certain first-order AODE. This result leads to a method for finding liouvillian solutions of certain first-order AODEs of positive genera in the case that their reduced forms are of genus zero (Section 4.2.3).

5. We transform the problem of solving first-order AODEs with liouvillian coefficients into the case of solving an AODE (4.1) by means of change of variables (Section 4.4).

The following is a short description of our future research.

1. **Study on the relation of the positive genera of first-order AODEs which are generated by putting a power transformation** (4.19) **into the ones of genus zero.** In Section 4.2, we have considered this problem but not yet to give an explicit relations of such genera. To attack this problem, we are working on related documents [8, 24, 27, 31].

2. **Keep focusing on the problem of determining liouvillian solutions of first-order ODEs** (4.7). This problem has been consulted in Section 4.1.3 and we will keep it going by focusing on the related works [4, 9, 53–55].

List of author's related publications

1. Nguyen T. D., Ngo L. X. C. (2021), "Rational liouvillian solutions of algebraic ordinary differential equations of order one", *Acta Mathematica Vietnamica*, 46 (4), pp. 689–700.

2. Nguyen T. D., Ngo L. X. C. (2023), "Liouvillian solutions of algebraic ordinary differential equations of order one of genus zero", *Journal of Systems Science and Complexity*, 36(2), pp. 884–893.

3. Nguyen T. D., Ngo L. X. C., "Liouvillian solutions of first-order algebraic ordinary differential equations", submitted.

4. Nguyen T. D. (2024), "Finding liouvillian solutions of first-order algebraic ordinary differential equations by change of variables", *Quy Nhon University Journal of Science*, 18(3), pp. 83–89.

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